

Two-component Abelian sandpile models

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In one-component Abelian sandpile models, the toppling probabilities are independent quantities. This is not the case in multicomponent models. The condition of associativity of the underlying Abelian algebras imposes nonlinear relations among the toppling probabilities. These relations are derived for the case of two-component quadratic Abelian algebras. We show that Abelian sandpile models with two conservation laws have only trivial avalanches.

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I. INTRODUCTION

Sandpile models are important toy models to understand self-organized criticality [1]. In an Abelian sandpile model, the toppling rules can be encoded in an Abelian algebra. This was pointed out by Dhar *et al.* [2–4]. The structure of these algebras [5] (see also [6]) is very simple (their physical relevance will be shown later in the text). They are defined by taking graphs (see Fig. 1) and attaching to each vertex a generator a_i of the algebra. All the generators commute with each other. Two vertices are connected by at most two links oriented in opposite directions. To each vertex “ i ” we attribute a polynomial relation which expresses a power of a_i (say n) as a polynomial in the generators attached to the sites reached by the outgoing arrows starting at i as well as a_i . The degree of the polynomial is at most equal to n . This implies, for example, that for the vertex “0” in Fig. 1 we have

$$a_0^n = P(a_0, a_1, a_2, a_3, a_4). \quad (1)$$

In the corresponding sandpile model, a_0^k is interpreted as having k grains of sand on the vertex 0. The coefficients in the polynomial $P(a_0, a_1, a_2, a_3, a_4)$ are non-negative and their sum is equal to 1. As we are going to see, they are going to be interpreted as probabilities.

A simple and very relevant example [1] is the case of a two-dimensional square lattice [coordinates (i, j)] with the Abelian algebra given by the relations

$$a_{i,j}^4 = a_{i+1,j} a_{i,j+1} a_{i-1,j} a_{i,j-1}, \quad [a_{i,j}, a_{i',j'}] = 0. \quad (2)$$

We did not specify the boundary conditions.

The sandpile model is defined by a stochastic process which gives the stationary state and the rules how the sand grains act. In continuous time and a lattice with N vertices

(sites) the time evolution of the system is given by a Hamiltonian H ,

$$H = \sum_{i=1}^N w_i (1 - a_i), \quad (3)$$

where the non-negative coefficients w_i are transition rates and we chose the unit of time by taking

$$\sum_{i=1}^N w_i = 1. \quad (4)$$

Hamiltonian (3) acts in an M -dimensional vector space given by all the independent M monomials in the generators a_i . There is a correspondence between each monomial and a configuration in the sandpile model, in which to each generator appearing in the monomial corresponds to sand grains on the respective sites. Their number equals the power of which the generator appears in the monomial

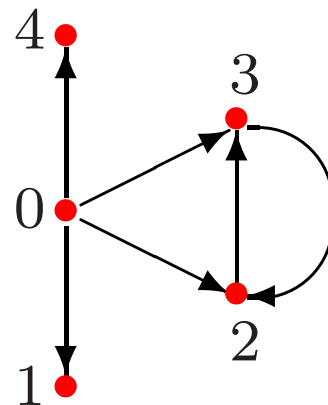


FIG. 1. (Color online) A typical graph on which one can define Abelian algebras useful for sandpile models. There are four oriented links leaving the vertex 0. The sites 2 and 3 are connected by two edges of opposite orientation.

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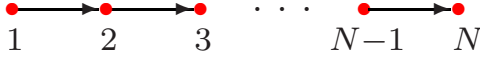


FIG. 2. (Color online) A one-dimensional acyclic directed lattice with N sites. All edges have the same orientation.

The action of H on the configuration space can be understood in the following way: take a configuration, in a unit of time, with a probability w_i a grain of sand is added at the site i . As a result this configuration goes to others, with probabilities given by the toppling rules (1).

The non-normalized probabilities $P_m(t)$ to find the system in a configuration “ m ” at the time t can be obtained from the master equation

$$\frac{dP(t)}{dt} = -HP(t), \quad (5)$$

where

$$P(t) = \sum_{m=1}^M P_m(t)W(m), \quad (6)$$

where $W(m)$ is the monomial corresponding to the configuration m . The stationary probability distribution function (PDF) will be denoted by $|0\rangle$ ($H|0\rangle=0$).

Except for positivity there are no supplementary constraints in relations (1). This implies that one can choose at will the toppling probabilities; however, as we are going to see this is *not* the case if one considers two-component sandpile models.

II. SANDPILE MODELS WITH TWO KINDS OF SAND

In two-component Abelian sandpile models, one assumes that one has two types of sand (say “ a ” and “ b ”). One adds with a probability u_i (v_i) a grain of sand a (b) to the site i . Grains topple if there are more than one, of any kind, on a given site. In the toppling process, the two types of sand mix and eventually transform into each other. We are going to consider the simplest model of this kind.

Consider a one-dimensional directed lattice (Fig. 2) with N sites [5]. To each site i we attach two generators a_i and b_i , all of them mutually commuting,

$$[a_i, a_{i'}] = [a_i, b_{i'}] = [b_i, b_{i'}] = 0 \quad \forall i, i' = 1, \dots, N. \quad (7)$$

For simplicity we consider quadratic algebras only. The most general quadratic relations involving nearest-neighbor interactions only are

$$\begin{aligned} a_i^2 &= \alpha_1 a_i a_{i+1} + \alpha_2 a_i b_{i+1} + \beta_1 b_i a_{i+1} + \beta_2 b_i b_{i+1} \\ &\quad + \xi_1 a_{i+1}^2 + \xi_2 b_{i+1}^2 + \xi_3 a_{i+1} b_{i+1}, \\ b_i^2 &= \gamma_1 a_i a_{i+1} + \gamma_2 a_i b_{i+1} + \delta_1 b_i a_{i+1} + \delta_2 b_i b_{i+1} \\ &\quad + \eta_1 a_{i+1}^2 + \eta_2 b_{i+1}^2 + \eta_3 a_{i+1} b_{i+1}, \\ a_i b_i &= \mu_1 a_i a_{i+1} + \mu_2 a_i b_{i+1} + \nu_1 b_i a_{i+1} + \nu_2 b_i b_{i+1} \\ &\quad + \zeta_1 a_{i+1}^2 + \zeta_2 b_{i+1}^2 + \zeta_3 a_{i+1} b_{i+1} \end{aligned} \quad (8)$$

for $i=1, \dots, N$, in which we take

$$a_{N+1} = b_{N+1} = 1. \quad (9)$$

This implies that the two types of sand may leave the system on the site N . The constants in Eq. (8) are positive probabilities, so that

$$\alpha + \beta + \xi = \gamma + \delta + \eta = \mu + \nu + \zeta = 1, \quad (10)$$

where we used the notation $\alpha = \sum_{i=1}^2 \alpha_i$, $\beta = \sum_{i=1}^2 \beta_i$, $\xi = \sum_{i=1}^3 \xi_i$, etc.

Relations (7)–(9) do not define yet an algebra since one has still to impose associativity (diamond conditions),

$$(a_i^2) b_i = a_i (a_i b_i), \quad a_i (b_i^2) = (a_i b_i) b_i. \quad (11)$$

Introducing Eq. (8) in Eq. (11) we find the following 12 relations:

$$\xi_i = \beta_i(\mu_i - \delta_i) + \nu_i(\nu_i - \alpha_i), \quad i = 1, 2,$$

$$\xi_3 = \beta_1(\mu_2 - \delta_2) + \beta_2(\mu_1 - \delta_1) + \nu_1(\nu_2 - \alpha_2) + \nu_2(\nu_1 - \alpha_1),$$

$$\eta_i = \gamma_i(\nu_i - \alpha_i) + \mu_i(\mu_i - \delta_i), \quad i = 1, 2,$$

$$\eta_3 = \gamma_1(\nu_2 - \alpha_2) + \gamma_2(\nu_1 - \alpha_1) + \mu_1(\mu_2 - \delta_2) + \mu_2(\mu_1 - \delta_1),$$

$$\zeta_i = \beta_i \gamma_i - \mu_i \nu_i, \quad i = 1, 2,$$

$$\zeta_3 = \beta_1 \gamma_2 + \beta_2 \gamma_1 - \mu_1 \nu_2 - \mu_2 \nu_1, \quad (12)$$

and

$$\beta(1 + \mu - \delta) = (1 - \nu)(1 + \nu - \alpha), \quad (13)$$

$$\gamma(1 + \nu - \alpha) = (1 - \mu)(1 + \mu - \delta), \quad (14)$$

$$\beta\gamma = (1 - \mu)(1 - \nu). \quad (15)$$

Out of the three relations (13)–(15) only two are independent since multiplying Eq. (13) with Eq. (14) one obtains Eq. (15).

As one can see from Eqs. (12)–(15) in the two-component case the various toppling probabilities are constrained by nonlinear relations. The vector space in which the Hamiltonian

$$H = \frac{1}{N} \sum_{i=1}^N (1 - u_i a_i - v_i b_i), \quad u_i + v_i = 1 \quad (16)$$

acts is made out of monomials in which either a_i , b_i , or 1 appears for a given site i . The physical interpretation of these monomials is obvious. If in the monomial a_i (b_i) appears, on the site i , one has a grain of sand of type a (b). If neither a_i nor b_i appears in the monomial, there is a vacancy on the site i .

It is easy to show the stationary state PDF is of product form,

$$|0\rangle = \prod_{i=1}^N \frac{(1 - \alpha)(1 - \delta) - \mu\nu + (1 + \mu - \delta)a_i + (1 + \nu - \alpha)b_i}{(2 - \alpha)(2 - \delta) - (1 - \mu)(1 - \nu)}. \quad (17)$$

An obvious question is if there is a solution of Eqs. (12)–(15) in which each type of sand is conserved separately. This would imply perhaps a new universality class of sandpile models. If one takes only α_1 , ξ_1 , δ_2 , η_2 , μ_2 , ν_1 , and ζ_3 as nonvanishing probabilities, in the bulk, the number of sand grains of types a and b is conserved separately. There are only two equivalent solutions of the diamond conditions in this case. One solution is

$$a_i^2 = \alpha a_i a_{i+1} + (1 - \alpha) a_{i+1}^2, \quad (18)$$

$$b_i^2 = b_i b_{i+1}, \quad (19)$$

$$a_i b_i = b_i a_{i+1}. \quad (20)$$

In the second solution, one exchanges the generators a_i with b_i .

The stationary PDF of the stochastic process (16) and (18)–(20) is

$$|0\rangle = \prod_{i=1}^N b_i. \quad (21)$$

There are only trivial avalanches in this case. If a grain of sand of type a hits the system, it leaves directly at the boundary because of Eq. (20), leaving the system unchanged. If a sand grain of type b hits the system, it also leaves it directly at the boundary because of Eq. (19). This is a surprising result.

We consider now a more general graph where the vertex i is linked by outgoing arrows to a number of sites which we label by index x . If one asks for two conservation laws and imposes the associativity condition (11), the generalization of the solution (18)–(20) reads

$$a_i^2 = \sum_x \alpha_x a_i a_x + \sum_{x,y} \xi_{x,y} a_x a_y, \quad (22)$$

$$b_i^2 = \sum_x \delta_x b_i b_x, \quad (23)$$

$$a_i b_i = \sum_x \nu_x b_i a_x, \quad (24)$$

where the toppling probabilities satisfy the relations $\sum_x \delta_x = \sum_x \nu_x = 1$, $\alpha_x \leq \nu_x$, and

$$\xi_{x,y} = \frac{1}{2} \{ \nu_x (\nu_y - \alpha_y) + \nu_y (\nu_x - \alpha_x) \}. \quad (25)$$

Notice that, similar to the one-dimensional case, only the grains of sand of type a topple. This has as consequence that for an arbitrary graph the stationary state of the system is an absorbing state, like Eq. (21), in which only particles b are present. The proof is straightforward. Only trivial avalanches can be obtained.

We have shown that using other representations of the algebra (18)–(20) which may define other sandpile models (see [5]), one obtains the same result: two conservation laws are compatible with trivial avalanches only. The system stays unchanged in the aftermath.

We have checked, only for quadratic algebras, that for a vertex connected by arrows to several vertices [there was

only one in example (8)], the associativity conditions (11) are incompatible with the existence of nontrivial avalanches in the sandpiles with two conservation laws. This is a surprising result. We have no proof of a similar statement for more general algebras (cubic, quartic, etc.). Simple examples like the generalization of Eq. (2) give the same result.

If we do not insist on two conservation laws, there are various solutions of the constraints (12)–(15) and therefore two-component sandpile models. They all share the property that during the toppling process, grains of sand of types a and b mutate (we have not identified a physical process which leads to such a phenomenon). We present a simple example in which we consider the Abelian algebra,

$$a_i^2 = (1 - \nu) a_i a_{i+1} + \frac{\nu}{1 + \mu} b_i a_{i+1} + \frac{\nu \phi}{1 + \mu} a_{i+1}^2 + \frac{\nu(\mu - \phi)}{1 + \mu} a_{i+1} b_{i+1}, \quad (26)$$

$$b_i^2 = (1 - \mu) b_i b_{i+1} + \phi(1 + \mu - 2\phi) a_{i+1} b_{i+1} + \phi^2 a_{i+1}^2 + (1 - \phi)(\mu - \phi) b_{i+1}^2, \quad (27)$$

$$a_i b_i = \phi a_i a_{i+1} + (1 - \phi) a_i b_{i+1}, \quad (28)$$

and Hamiltonian (16) with $u_i = v_i = 1/2$. This implies that we take the stationary state

$$|0\rangle = \prod_{i=1}^N \frac{\mu \nu + (1 + \mu) a_i + \nu b_i}{(1 + \mu)(1 + \nu)} \quad (29)$$

and add with equal probability a grain of sand of types a and b on the first site. Notice that $|0\rangle$ is independent on ϕ .

We are interested to know what is the probability $P_a(T)$ to have an avalanche ending with a grain a and having a duration T . The duration of the avalanche, in this one-dimensional case, is the number of sites where the topplings occur. $P_b(T)$ has a similar meaning. In [5] it was shown that in the case of the one-component model which is obtained by considering only Eq. (27) in which one takes $\phi=0$, at large values of T , one has

$$P_b(T) \sim T^{-3/2}. \quad (30)$$

This means that for the one-component case the avalanches are in the random walker universality class [7]. In Fig. 3 we present the results of Monte Carlo simulations for the model given by Eqs. (26)–(28) in which we have taken $\mu = \nu = 1$ and $\phi = 1/2$. We observe that for large values of T ,

$$P_a(T) \sim T^{-3/2}, \quad P_b(T) \sim T^{-3/2}, \quad (31)$$

mutations between sand grains of types a and b do not change the universality class. This stays valid for the whole parameter space except for the boundaries $\phi=0$ and $\phi=\mu$.

If in Eqs. (26)–(28) one takes $\phi=0$, $P_a(T)$ decreases exponentially and $P_b(T)$ has an algebraic falloff (30). This can be easily shown using the algebra (26)–(28) and the recurrence relations for the toppling probabilities which follow. Conversely, if $\phi=\mu$, $P_b(T)$ decays exponentially and $P_a(T)$ has the algebraic falloff.

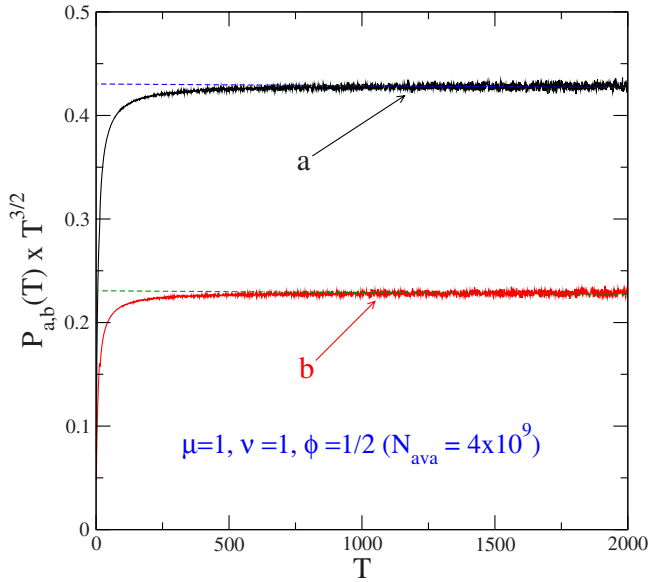


FIG. 3. (Color online) Probability distribution functions $P_a(T)$ and $P_b(T)$ to have avalanches with duration T ending with grains of sand of types a and b , respectively, obtained in the model given by Eqs. (26)–(28). The values $\mu=\nu=1$ and $\phi=1/2$ were used for the parameters of the model. 4×10^9 avalanches were observed in the Monte Carlo simulations. In the figure $P_a(T)$ and $P_b(T)$ are multiplied by $T^{3/2}$ in order to show their large- T behavior.

III. CONCLUSIONS

The main result of this Brief Report is that if one is interested in two-component Abelian sandpile models, the toppling probabilities are not arbitrary. They have to satisfy non-linear relations coming from the condition that the algebra is associative. These constraints do not exist in the one-component models. An unexpected result is that, at least for the case of quadratic algebras, the coexistence of two con-

servation laws in the bulk (one for each component), and of nontrivial avalanches, is impossible.

We have also shown in an example of a one-dimensional directed model that once we allow the two components to mutate in each other during the toppling process, the one-component and the two-component models belong to the same universality class. We believe that our conclusions apply to any multicomponent models.

We would like to mention a possible extension of the quadratic algebra (8). If we omit the last of the three equations (8), two grains of sand, one of type a and the other one of type b , do not topple. As a result the stationary state is a linear combination of 4^N instead of 3^N states. If we are not interested in mutations ($a \leftrightarrow b$), one has a direct product of two algebras. One contains only a_i generators [like Eq. (18)] and the other one contains only b_i generators. One gets avalanches in which grains of types a and b do not mix. If one considers the possibility of mutations the situation is different. Take for example the algebra defined by Eqs. (26) and (27) [we have omitted Eq. (28)], the total number of “grains” is not conserved since $a_{i+1}b_{i+1}$ has to be looked up as a new kind of grain. As a result, the PDFs of the duration of avalanches ending in a grain of sand of type a , b , or ab have an exponential falloff. We have also checked that if we change the algebra by allowing mutations, while conserving the number of sand grains, the avalanches belong to the random walker universality class [7].

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